CHROMATIC POLYNOMIAL OF GRAPHS

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Abstract:

This paper studies various results on vertex colorings of simple connected graphs, chromatic number, chromatic polynomials and some Algebraic properties of chromatic polynomials. Results were obtained on the roots of chromatic polynomials of simple connected graphs based on Read's conjecture. The chromatic number of every graph is the minimum number of colors to properly color the graph. Chromatic polynomial of a graph is a polynomial in integer and the leading coefficient of chromatic polynomial of a graph of order *n* and size *m* is always1, whose coefficient alternate in sign. Through the application of famous graph theorem (the hand shaking lemma) by whiskey which states that: "the order of a graph twice its size". Hence, every graph has a chromatic polynomial but not all polynomials are chromatic. For example, the polynomial $\lambda^5 - 11 \lambda^4 + 14 \lambda^3 - 6 \lambda^2 + 2 \lambda$ is a polynomial for a graph on five vertices and eleven edges which does not exists. Because the maximum number size for a graph of order five is ten. The paper gave the property of chromatic polynomial of isomorphic graphs and bipartite graphs.

Keywords: Adjacent Vertices ,isomorphic graphs Chromatic Number, Chromatic Polynomials

1. Introduction

A graph is a pair $G=(V,E)$,where V is the set of vertices and E is the set of edges. Here we consider only nontrivial, finite, connected, undirected graph without loops or multiple edges. Vertices are sometimes called points or nodes, while edges are sometimes referred to as lines or links. Each edge $\{u, v\}$ of V is commonly denoted by uv or vu. If $e = uv$, then the edge e is said to join u and v. The number of vertices in a graph G is the order of G and the number of edges is the size of G. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Bondy [1].

A graph of order 1 is called a trivial graph and so a nontrivial graph has two or more vertices. A graph of size 0 is an empty graph and so a nonempty graph has one or more edges. Graphs are typically represented by diagrams in which each vertex is represented by a point or small circle (open or solid) and each edge is represented by a line segment or curve joining the corresponding small circles [2]. A diagram that represents a graph G is referred to as the graph G itself and the small circles and lines representing the vertices and edges of G .

Isomorphic Graphs: Two graphs G_1 and G_2 are said to be isomorphic if −Their number of components (vertices and edges) are same. Their edge connectivity is retained.

2. Vertex Colorings

The vertices of a graph G can also be classified using colorings. These colorings tell that certain vertices have a common property (or that they are similar in some aspect), if they share the same color. The problem is, if we have n colors, then we have to find a way for coloring vertices such that no two adjacent vertices have the same color. There exists some other graph coloring problems also, for example, Edge Coloring and Face coloring. In edge coloring, not a single vertex is connected to two edges which are having same color. [10]. However the graph *G* contains three mutually adjacent vertices and hence is not 2-colorable. Thus *G* is three chromatic [3,6].

Definition 2.1: Proper Vertex Coloring

A proper vertex coloring of a graph *G* is a vertex coloring such that the end points of each edge are assigned two different colors [9].

Definition 2.2: Chromatic Polynomials(Chronomials) **[4]**

During the period that the Four Color Problem was unsolved, which spanned more than a century, many approaches were introduced with the hopes that they would lead to a solution of this famous problem. Birkhoff (1912) defined a function *P* (*M,* λ) that gives the number of proper λ-colorings of a map *M* for a positive integer λ. As we will see, *P* (*M,* λ) is a polynomial in λ for every map *M* and is called the chromatic polynomial of *M*. Consequently, if it could be verified that *P* (*M*, 4) > 0 for every map *M*, then this would have established the truth of the Four Color Conjecture. Renewed interest in chromatic polynomials of graphs occurred. Read (1968) wrote a survey paper on chromatic polynomials. For a graph *G* and a positive integer λ, the number of different proper λ-colorings of *G* is denoted by *P* (*G*, λ) and is called the chromatic polynomial of *G*. [5]

Definition 2.3:

For two graphs G and H, or $h_1 = h_2$ and $g_1 g_2 \varepsilon E(G)$,their Cartesian product $G \times H$ has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is joined (g_2, h_2) iff $g_1 = g_2$ and $h_1 h_2 \varepsilon E(H)$

Definition 2.4:

Mongolian tent as a graph obtained from $P_m \times P_n$, n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex.

Theorem 2.5: P(C4, λ) = $(\lambda - 1)^4 + (\lambda - 1)$. [11]

Theorem 2.6: The lead coefficient of P (*G,* λ) is always 1. [11]

Results 2.5: Some Algebraic Properties of Chromatic Polynomials

Let *G* be a graph and λ be the set of colors to color *G*

1. The lead coefficient of P (*G,* λ) is always1.

   a) The coefficient of λ^{n-1}in P (*G,* λ) is the negative of the number of edges.

2. The constant term, i.e. the coefficient of 1 in P (G, λ) is always zero.
3. The coefficient of λ in P(*G,*λ) is non-zero if and only if *G* is connected.
4. The coefficients of the chromatic polynomial alternate in sign.
5. The chromatic polynomial has no real root greater than *n*–1. Every two trees of the same order are chromatically equivalent. It is not known under what conditions two graphs are chromatically equivalent in general [5,7,8].

3. Chromatic Polynomial of Isomorphic Graphs

Result 3.1: The Chromatic Polynomial of Isomorphic Graphs are same.

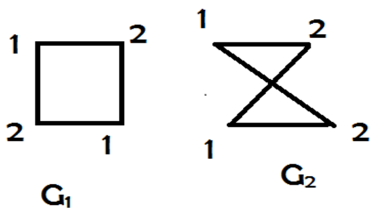
An isomorphism of graphs *G* and *H* is a bijection between the vertex sets of *G* and *H*

such that any two vertices *u* and *v* of *G* are adjacent in *G* if and only if *f*(*u*) and *f*(*v*) are

adjacentin *H*. From the definition we observe that all the vertices and edges of two

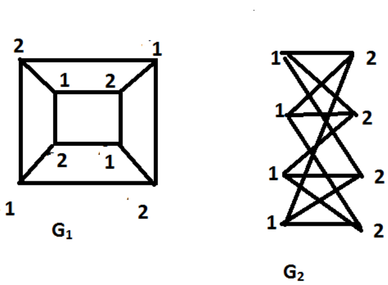
isomorphic graphs preserves same property, hence the chromatic polynomials of isomorphic graphs are same.

Example: The following graphs G_1 and G_2 are isomorphic graphs with 2- colorable.



The chromatic polynomial of these graphs is defined as

$$P(G_1, \lambda) = P(G_2, \lambda) = \left[\lambda(\lambda-1)\right]^2$$

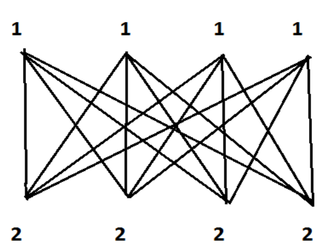


The chromatic polynomial of these graphs is defined as

$$P(G_1, \lambda) = P(G_2, \lambda) = \left[\lambda(\lambda-1)\right]^4$$

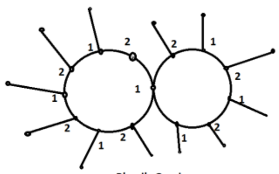
Result 3.2:

The chromatic polynomial of bipartite graphs is P(Km, Kn) = λ^n



Result 3.3:

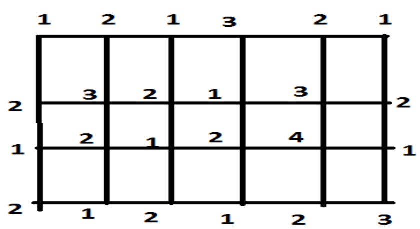
The chromatic polynomial of bicyclic graphs is P(Km, Kn) = $\left[\lambda(\lambda-1)\right]^{n}$



Bicyclic Graph

Theorem 2.4:

For $p_m \times p_n$, the chromatic polynomial is $P(P_m \times P_n) = \lambda^9 (\lambda-1)^{10} (\lambda-2)^3 (\lambda-3)$.



$P_5 X P_3$

4. Conclusion:

A way of finding the minimum number of colors to color a graph is by means of vertex coloring. The use of vertex coloring also assists in ensuring this great achievement of stable networks. The chromatic polynomial of any Graph exists, we don't know yet under what condition two graphs (except isomorphic graphs) will have the same chromatic polynomial in general.

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