$(\alpha\text{-}\psi)_s$ -Geraghty Contractions on b-Generalized Metric Spaces

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Abstract
The notion of $(\alpha\text{-}\psi)_s$ -Geraghty contraction is introduced in the context of b-generalized metric space and we ensure the existence and uniquness of fixed point for this contraction. As a consequence, we obtain a common fixed point theorem. In the process, we improve and generalize recent fixed point results in b-metric spaces and generalized metric spaces established by Faraji et al. (Axioms, 2019) and Asadi et al. (Journal of Inequalities and Applications, 2014) respectively.

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1. Introduction

Fixed point theory play a vital role in solving various problem of nonlinear analysis which can be modeled in an abstract form of nonlinear equation of type $Tx = x$ where T is a selfmap defined on some abstract space. Banach contraction principle is the most significant result of fixed point theory which states that every contraction has a unique fixed point in the set up of a complete metric space. In an attempt to extend this contraction principle several authors generalized the usual metric space in different way.

One of the generalization of metric space was introduced by Bakhtin [1] in 1989 and he called such a space as b-metric space which is defined as follows.

Definition 1.1 Let X be a nonempty set, $s \geq 1$ be a given real number and let $d: X \times X \to [0,\infty)$ be a mapping such that for all $x, y, z \in X$ satisfying the following axioms

(a) $d(x,y) = 0$ if and only if $x = y$;

(b) $d(x,y) = d(y,x)$;

(c) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then d is called a b-metric and the space (X,d) is called a b-metric space. For more details about this space, one can refer Czerwik [10].

In 2000, Branciari [2] introduced another generalization of metric space where he replaced the triangle inequality in metric space with the quadrilateral inequality and named this space as generalized metric space. As, the generalized metric need not be continuous and the respective topology is not necessarily to be Hausdorff, therfore this space continously drawing the attention of the researchers

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working in the area of fixed point. Recently, Roshan et al. [6] defined a new space namely b- generalized metric space by inspiring to the idea of Bakhtin [1].

Here, we define the notion of $(\alpha\text{-}\psi)_s$ -Geraghty contractions in the set up of b-generalized metric space and obtain some new results as an extension of already existing fixed point as well as common fixed point theorems. More specifically, our results will improve and generalize recent fixed point results in b-metric spaces and generalized metric spaces established by Faraji et al. [3] and Asadi et al. [4] respectively.

2. Preliminaries

The present section deals with some basic definitions, standard notations and auxillary results which will be required in the sequel. Throughout, the symbol $\mathbb{N}$ and $\mathbb{R}$ are used to represent the set of natural numbers and set of real numbers respectively.

The following is the definition of b-generalized metric space which is given in [5, 6].

Definition 2.1 Let X be a nonempty set, $s \geq 1$ be a given real number. Then a mapping $d: X \times X \rightarrow [0, \infty)$ is said to be a b-generalized metric if it satisfy the following axioms for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y,

(a) $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ (symmetric);

(c) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$.

The space X equipped with this d is called b-generalized metric space and abbreviated as bGMS.

It is clear from the above definition that b-metric space and generalized metric space are b-generalized metric space but the converse is not necessarily true. This can be seen in the following example, which is a modification and correction to Example 1.7 of [5].

Example 2.2 Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $B = \{2,3\}$ and $X = A \cup B$. Define $d: X \times X \rightarrow [0, \infty)$ such that

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 2, & \text{if } x, y \in A \\ \frac{1}{2(n+1)}, & \text{if } x \in A, y \in B \text{ or } x \in B, y \in A \\ 1, & \text{otherwise.} \end{cases}$$

Then, it is easy to check that (X, d) is a b-generalized metric space with $s = 2 > 1$ but (X, d) is not a generalized metric space since

$$\begin{aligned} d\left(\frac{1}{3}, \frac{1}{4}\right) = 2 > \ & 1 + \frac{1}{n+1} \\ & = \frac{1}{2(n+1)} + 1 + \frac{1}{2(n+1)} \\ & = d\left(\frac{1}{3}, 2\right) + d(2,3) + d\left(3, \frac{1}{4}\right). \end{aligned}$$

Also, it is not possible to find any $s \geq 1$ such that $d(x,y) \leq s[d(x,z)+d(z,y)]$ for all $x,y,z \in X$, therefore (X,d) is not a b-metric space.

Defintion 2.3 Let $\{x_n\}$ be a sequence in a b-generalized metric space (X,d). Then

(a) $\{x_n\}$ is said to converges to a point x in X, if for a given $\epsilon > 0$, there exists a natural number say m such that $d(x_n,x) < \epsilon$ for all $n \geq m$. This can be re-written as $d(x_n,x) \to 0$ as $n \to \infty$.

(b) $\{x_n\}$ is said to be Cauchy sequence, if for a given $\epsilon > 0$, there exists a natural number say N such that $d(x_n,x_m) < \epsilon$ for all $n,m \geq N$. In other words, $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

(c) The space (X,d) is said to be complete if Cauchy sequence in X converges to some point of X.

In Example 2.2, it is important to note that $d\left(\frac{1}{n},2\right) \to 0$ and $d\left(\frac{1}{n},3\right) \to 0$ as $n \to \infty$, therefore the sequence $\left\{\frac{1}{n}\right\}$ does not converges to unique limit. Also, the metric d is not continuous since the sequence $\left\{\frac{1}{n}\right\}$ converges to $x=2$ but $d\left(\frac{1}{n},3\right) \to 0 \neq 1 = d(2,3)$. Thus to overcame this problem, in general, we required the following lemmas to prove our main result.

Lemma 2.4 [6] Let (X,d) be a b-generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.

Samet et al. [7] introduced the notion of α-admissible mappings as follows.

Definition 2.5 Let X be a nonempty set and let $T: X \to X$ and $\alpha: X \times X \to [0,\infty)$ be mappings. Then T is called α-admissible if for all $x,y \in X$, we have

$$\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1.$$

Recently, Karapınar et al. [8] defined the notion of triangular α-admissible mappings as follows.

Definition 2.6 Let X be a nonempty and let $T: X \to X$ and $\alpha: X \times X \to \mathbb{R}$ be mappings. Then T is called triangular α-admissible if

(i) $x,y \in X$, $\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$;

(ii) $x,y,z \in X$, $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1 \implies \alpha(x,y) \geq 1$.

Lemma 2.7 [8] Let $T: X \to X$ be a triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.

3. $(\alpha$-$\psi)_s$-Geraghty contraction

Let $\mathcal{F}$ be the class of functions $\beta: [0,\infty) \to \left[0,\frac{1}{s}\right)$ which satisfy the condition

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies } \lim_{n\to\infty} t_n = 0. \tag{3.1}$$

Definition 3.1 Let (X,d) be a b-generalized metric space with coefficient $s \geq 1$. Then the self map T defined on X is called $(\alpha\text{-}\psi)_s$ -Geraghty contraction, if for all $x, y \in X$, we have

$$\alpha(x,y)\psi(sd(Tx,Ty)) \leq \beta\left(\psi(d(x,y))\right)\psi(d(x,y)), \tag{3.2}$$

where $\alpha: X \times X \to [0,\infty)$, $\beta \in \mathcal{F}$ and $\psi: [0,\infty) \to [0,\infty)$ is non-decreasing, continuous and $\psi(0) = 0$.

Theorem 3.2 Let (X,d) be a complete b-generalized metric space with coefficient $s \geq 1$ and T be a self map of X such that T is $(\alpha\text{-}\psi)_s$ -Geraghty contraction and satisfy the following

(i) T is triangular α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;

(iii) T is continuous.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof: Take a point $x_0 \in X$ and define a sequence $\{x_n\}$ as

$$x_{n+1} = Tx_n = T^n x_0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

If $x_n = x_{n+1}$ for some n, then for such n, x_n itself is a fixed point of T and hence the result is vacuously true. Let us assume $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. As T is triangular α-aadmissible, therefore the hypothesis (ii) implies that

$$\alpha(x_1, x_2) = \alpha(x_1, Tx_1) \geq 1 \quad \text{and} \quad \alpha(x_1, x_3) = \alpha(x_1, T^2x_1) \geq 1.$$

By Lemma 2.7, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_n, x_{n+2}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Also, we can obtain $\alpha(x_n, x_{n+m}) \geq 1$ for each $m, n \in \mathbb{N}$.

Now, we shall prove that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Using (3.3) and nondecreasing property of ψ, we have

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \psi(sd(x_n, x_{n+1})) \\ &\leq \alpha(x_{n-1}, x_n)\psi(sd(Tx_{n-1}, Tx_n)).\end{aligned}$$

As T is $(\alpha\text{-}\psi)_s$ -Geraghty contraction, it follows that

$$\psi(d(x_n, x_{n+1})) \leq \beta\left(\psi(d(x_{n-1}, x_n))\right)\psi(d(x_{n-1}, x_n)). \tag{3.4}$$

Due to the fact that $\beta \in \mathcal{F}$, Equation (3.4) implies

$$\psi(d(x_n, x_{n+1})) < \frac{1}{s}\psi(d(x_{n-1}, x_n)), \quad n \in \mathbb{N}.$$

Since ψ is nondecreasing, we get that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \quad \text{for each } n \in \mathbb{N}.$$

Thus, we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and decreasing. So, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Assume that $r > 0$. Then, (3.4) yields that

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta\left(\psi(d(x_n, x_{n+1}))\right) < \frac{1}{s}.$$

Now, applying $n \to \infty$, we obtain $\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta\left(\psi\big(d(x_n, x_{n+1})\big)\right) < \frac{1}{s}$, which yields that $\limsup_{n\to\infty} \beta\left(\psi\big(d(x_n, x_{n+1})\big)\right) = \frac{1}{s}$. Thus, we have

$$\lim_{n\to\infty} \psi\big(d(x_n, x_{n+1})\big) = 0.$$

On account of the continuity of ψ, it follows that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.5}$$

Similarly, we prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. On account of the Equation (3.3) and (3.2), we get

$$\begin{aligned} \psi\big(d(x_n, x_{n+2})\big) &\leq \alpha(x_{n-1}, x_{n+1})\psi\big(sd(Tx_{n-1}, Tx_{n+1})\big) \\ &\leq \beta\left(\psi\big(d(x_{n-1}, x_{n+1})\big)\right)\psi\big(d(x_{n-1}, x_{n+1})\big) \\ &< \frac{1}{s}\psi\big(d(x_{n-1}, x_{n+1})\big), \quad n \in \mathbb{N}. \end{aligned} \tag{3.6}$$

Since ψ is nondecreasing, we derive

$$d(x_n, x_{n+2}) < d(x_{n-1}, x_{n+1}), \qquad \text{for each } n \in \mathbb{N}.$$

Thus, we observe that the sequence $\{d(x_{n-1}, x_{n+1})\}$ is nonnegative and decreasing. Therefore, there exists a constant $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = r$. We have to show that $r = 0$. On contrary, assume that $r > 0$. Then, (3.6) gives

$$\frac{\psi\big(d(x_n, x_{n+2})\big)}{\psi\big(d(x_{n-1}, x_{n+1})\big)} \leq \beta\left(\psi\big(d(x_{n-1}, x_{n+1})\big)\right) < \frac{1}{s}.$$

Both side taking the limit $n \to \infty$, we have $\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta\left(\psi\big(d(x_{n-1}, x_{n+1})\big)\right) < \frac{1}{s}$, which yields that $\limsup_{n\to\infty} \beta\left(\psi\big(d(x_{n-1}, x_{n+1})\big)\right) = \frac{1}{s}$. Hence

$$\lim_{n\to\infty} \psi\big(d(x_{n-1}, x_{n+1})\big) = 0.$$

Again using the continuity of ψ we can have

$$r = \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 0 = \lim_{n\to\infty} d(x_n, x_{n+2}). \tag{3.7}$$

We assert that the sequence $\{x_n\}$ has distinct terms that is $x_n \neq x_m$ for $n \neq m$. Suppose, on contrary, that $x_n = x_m$ for some $m < n$. Then

$$\begin{aligned} \psi\big(d(x_m, x_{m+1})\big) &= \psi\big(d(x_n, x_{n+1})\big) \\ &\leq \psi\big(sd(x_n, x_{n+1})\big) \quad (\psi \text{ is nondecreasing and } s \geq 1) \\ &\leq \beta\left(\psi\big(d(x_{n-1}, x_n)\big)\right)\psi\big(d(x_{n-1}, x_n)\big) \\ &< \psi\big(d(x_{n-1}, x_n)\big) \\ &\leq \psi^{n-m}\big(d(x_m, x_{m+1})\big) \\ &< \psi\big(d(x_m, x_{m+1})\big). \end{aligned}$$

This is a contradiction. Hence, all terms of the sequence $\{x_n\}$ are distinct.

We have to prove the sequence $\{x_n\}$ is Cauchy. On contrary, we assume that $\{x_n\}$ is not a Cauchy sequence then

$$\limsup_{n,m\to\infty} d(x_n, x_m) \neq 0. \qquad (3.8)$$

Here we need to examine two possible cases for $d(x_n, x_m)$ which are as follows. Without loss of generality we take $m > n$.

If $m - n$ is odd, then

$$d(x_n, x_m) \leq s\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m)\}.$$

This can be re-written as

$$d(x_n, x_m) - s\, d(x_n, x_{n+1}) - sd(x_{m+1}, x_m) \leq sd(x_{n+1}, x_{m+1}) = sd(Tx_n, Tx_m).$$

As ψ is nondecreasing, then by applying ψ on both side of the above equation and using the condition $\alpha(x_n, x_m) \geq 1$, we have

$$\begin{aligned}\psi\big(d(x_n, x_m) - sd(x_n, x_{n+1}) - sd(x_{m+1}, x_m)\big) &\leq \psi\big(sd(Tx_n, Tx_m)\big)\\ &\leq \alpha(x_n, x_m)\psi\big(sd(Tx_n, Tx_m)\big)\\ &\leq \beta\Big(\psi\big(d(x_n, x_m)\big)\Big)\psi\big(d(x_n, x_m)\big).\end{aligned}$$

Letting upper limit as $m, n \to \infty$, we deduce that

$$\begin{aligned}&\limsup_{m,n\to\infty} \psi\big(d(x_n, x_m) - sd(x_n, x_{n+1}) - sd(x_{m+1}, x_m)\big)\\ &\leq \limsup_{m,n\to\infty} \beta\Big(\psi\big(d(x_n, x_m)\big)\Big) \limsup_{m,n\to\infty} \psi\big(d(x_n, x_m)\big).\end{aligned}$$

Equation (3.5), (3.8) and the continuity of ψ imply

$$\frac{1}{s} \leq 1 \leq \limsup_{m,n\to\infty} \beta\Big(\psi\big(d(x_n, x_m)\big)\Big) < \frac{1}{s},$$

which yields that $\limsup_{m,n\to\infty} \beta\Big(\psi\big(d(x_n, x_m)\big)\Big) = \frac{1}{s}$. As $\beta \in \mathcal{F}$, thus $\lim_{m,n\to\infty} d(x_n, x_m) = 0$.

If $m - n$ is even, then

$$d(x_n, x_m) \leq s\{d(x_n, x_{n+2}) + d(x_{n+2}, x_{m+2}) + d(x_{m+2}, x_m)\},$$

which can be re-written as

$$d(x_n, x_m) - sd(x_n, x_{n+2}) - sd(x_{m+2}, x_m) \leq sd(Tx_{n+1}, Tx_{m+1}).$$

Applying ψ on both side, we obtain

$$\psi\big(d(x_n, x_m) - sd(x_n, x_{n+2}) - sd(x_{m+2}, x_m)\big) \leq \psi\big(sd(Tx_{n+1}, Tx_{m+1})\big).$$

As T is triangular α-admissible and T is $(\alpha$-$\psi)_s$ -Geraghty contraction. Therefore

$$\begin{aligned}\psi\big(d(x_n, x_m) - sd(x_n, x_{n+2}) - sd(x_{m+2}, x_m)\big) &\leq \alpha(x_{n+1}, x_{m+1})\psi\big(sd(Tx_{n+1}, Tx_{m+1})\big)\\ &\leq \beta\Big(\psi\big(d(x_{n+1}, x_{m+1})\big)\Big)\psi\big(d(x_{n+1}, x_{m+1})\big).\end{aligned}$$

Applying the upper limit as $m,n \to \infty$ and using (3.7), we get that

$$\limsup_{m,n\to\infty,} \psi\big(d(x_n,x_m)\big) \le \limsup_{m,n\to\infty} \beta\left(\psi\big(d(x_{n+1},x_{m+1})\big)\right) \limsup_{m,n\to\infty} \psi\big(d(x_{n+1},x_{m+1})\big).$$

So, by the continuity of ψ, we observe

$$\frac{1}{s} \le 1 \le \limsup_{m,n\to\infty} \beta\left(\psi\big(d(x_{n+1},x_{m+1})\big)\right) < \frac{1}{s},$$

which yields that $\limsup_{m,n\to\infty} \beta\left(\psi\big(d(x_{n+1},x_{m+1})\big)\right) = \frac{1}{s}$. Thus, we conclude that $\lim_{m,n\to\infty} d(x_{n+1},x_{m+1}) = 0$.

Hence in both the cases we get a contradiction to the hypothesis that $\{x_n\}$ is not a Cauchy sequence. So, the sequence $\{x_n\}$ is Cauchy. Since X is complete, then $\{x_n\}$ converges to some point say $x^* \in X$ that is $\lim_{n\to\infty} d(x_n,x^*) = 0$. As T is continuous, thus $\lim_{n\to\infty} d(x_{n+1},Tx^*) = 0$. Hence, Lemma 2.4 implies that x^* is a fixed point of T.

Remark 3.3 (1) In Theorem 3.2, if we take an assumption that there exists a subsequence $\{x_{n_k}\}$ to the given sequence $\{x_n\}$ with $\alpha\big(x_{n_k},x\big) \ge 1$ for all k, where $\{x_n\}$ is a convergent sequence converging to x, then continuity of T can be relaxed.

(2) If the map $\alpha: X \times X \to [0,\infty)$ satisfy the condition that $\alpha(x,y) \ge 1$ for all $x,y \in F(T)$, where $F(T)$ is the set of fixed points of T, then Theorem 3.2 will ensure the uniqueness of fixed point as well.

Now, we obtain a fixed point theorem for generalized ψ_s-Geraghty contraction.

Theorem 3.4 Let (X,d) be a complete b-generalized metric space with coefficient $s \ge 1$ and T be a self map of X such that T is generalized ψ_s-Geraghty contraction that is T satisfy the inequality for all $x,y \in X$

$$\psi\big(sd(Tx,Ty)\big) \le \beta\left(\psi\big(M(x,y)\big)\right)\psi\big(M(x,y)\big), \tag{3.9}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$, $\alpha: X \times X \to [0,\infty)$, $\beta \in \mathcal{F}$ and $\psi: [0,\infty) \to [0,\infty)$ is non-decreasing, continuous and $\psi(0) = 0$. Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof: Let $x_0 \in X$ be an arbitrary point. Construct a sequence $\{x_n\}$ as

$$x_{n+1} = Tx_n = T^n x_0 \qquad \text{for all } n \ge 0.$$

If there exists $n \ge 0$ such that $x_n = x_{n+1}$, then x_n itself is a fixed point of T and hence the result is true. Let us assume $x_n \ne x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$.

Firstly, we shall prove that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Since ψ is nondecreasing, we have

$$\begin{aligned} \psi\big(d(x_n,x_{n+1})\big) &\le \psi\big(sd(x_n,x_{n+1})\big) \\ &= \psi\big(sd(Tx_{n-1},Tx_n)\big). \end{aligned}$$

As T is generalized ψ_s-Geraghty contraction, it follows that

$$\psi\big(d(x_n, x_{n+1})\big) \leq \beta\left(\psi\big(M(x_{n-1}, x_n)\big)\right)\psi\big(M(x_{n-1}, x_n)\big). \quad (3.10)$$

Where $M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}$

$= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$.

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then by Equation (3.10), we arrived at contradiction. Hence

$$M(x_{n-1}, x_n) = d(x_{n-1}, x_n). \quad (3.11)$$

Due to the fact that $\beta \in \mathcal{F}$, Equation (3.10) implies

$$\begin{aligned}\psi\big(d(x_n, x_{n+1})\big) &< \tfrac{1}{s}\psi\big(d(x_{n-1}, x_n)\big)\\ &< \psi\big(d(x_{n-1}, x_n)\big), \quad n \in \mathbb{N}.\end{aligned}$$

Since ψ is nondecreasing, we get that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \quad \text{for each } n \in \mathbb{N}.$$

Thus, we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and decreasing. So, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Assume that $r > 0$. Then, (3.10) and (3.11) yields that

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta\left(\psi\big(d(x_{n-1}, x_n)\big)\right) < \frac{1}{s}.$$

Now, applying $n \to \infty$, we obtain $\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta\left(\psi\big(d(x_{n-1}, x_n)\big)\right) < \frac{1}{s}$, which yields that $\limsup_{n\to\infty} \beta\left(\psi\big(d(x_{n-1}, x_n)\big)\right) = \frac{1}{s}$. Thus, we have

$$\lim_{n\to\infty} \psi\big(d(x_{n-1}, x_n)\big) = 0.$$

On account of the continuity of ψ, it follows that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \quad (3.12)$$

We assert that the sequence $\{x_n\}$ has distinct terms that is $x_n \neq x_m$ for $n \neq m$. Suppose, on contrary, that $x_n = x_m$ for some $m < n$. Then

$$\begin{aligned}\psi\big(d(x_m, x_{m+1})\big) &= \psi\big(d(x_n, x_{n+1})\big)\\ &\leq \psi\big(sd(x_n, x_{n+1})\big) \quad (\psi \text{ is nondecreasing and } s \geq 1)\\ &\leq \beta\left(\psi\big(M(x_{n-1}, x_n)\big)\right)\psi\big(M(x_{n-1}, x_n)\big)\\ &< \psi\big(d(x_{n-1}, x_n)\big)\\ &\leq \psi^{n-m}\big(d(x_m, x_{m+1})\big)\\ &< \psi\big(d(x_m, x_{m+1})\big).\end{aligned}$$

This is a contradiction. Hence, all terms of the sequence $\{x_n\}$ are distinct.

We have to prove that the sequence $\{x_n\}$ is Cauchy. On contrary, we assume that $\{x_n\}$ is not a Cauchy sequence then there exists $\epsilon > 0$ and two sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \quad \text{and} \quad d(x_{m_k}, x_{n_k-2}) < \epsilon. \tag{3.13}$$

Using b-quadrilateral inequality, we have

$$d(x_{m_k}, x_{n_k}) \leq s\{d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})\}.$$

This can be re-written as

$$d(x_{m_k}, x_{n_k}) - s\, d(x_{m_k}, x_{m_k+1}) - sd(x_{n_k-1}, x_{n_k}) \leq sd(x_{m_k+1}, x_{n_k-1})$$
$$= sd(Tx_{m_k}, Tx_{n_k-2}).$$

Now applying ψ on both side of the above inequality, we have

$$\psi\left(d(x_{m_k}, x_{n_k}) - s\, d(x_{m_k}, x_{m_k+1}) - sd(x_{n_k-1}, x_{n_k})\right) \leq \psi\left(sd(Tx_{m_k}, Tx_{n_k-2})\right)$$
$$\leq \beta\left(\psi\left(M(x_{m_k}, x_{n_k-2})\right)\right)\psi\left(M(x_{n_k-2}, x_{n_k-1})\right).$$

Taking upper limit as $k \to \infty$, using (3.12) and the continuity of ψ, we get that

$$\lim_{k\to\infty}\sup\psi\left(d(x_{m_k}, x_{n_k})\right) \leq \lim_{k\to\infty}\sup\beta\left(\psi\left(M(x_{m_k}, x_{n_k-2})\right)\right)\lim_{k\to\infty}\sup\psi\left(M(x_{m_k}, x_{n_k-2})\right), \tag{3.14}$$

where

$$\lim_{k\to\infty}\sup\psi\left(M(x_{m_k}, x_{n_k-2})\right) = \lim_{k\to\infty}\sup\max\{d(x_{m_k}, x_{n_k-2}), d(x_{m_k}, x_{m_k+1}),\ d(x_{m_k+1}, x_{n_k-2})\}$$
$$\leq \lim_{k\to\infty}\sup d(x_{m_k}, x_{n_k-2}).$$

Since $\beta \in \mathcal{F}$, then using (3.13), it follows from (3.14) that

$$\frac{1}{s} \leq \lim_{k\to\infty}\sup\beta\left(\psi\left(M(x_{m_k}, x_{n_k-2})\right)\right) < \frac{1}{s}.$$

Which yields that $\lim_{k\to\infty}\sup\beta\left(\psi\left(M(x_{m_k}, x_{n_k-2})\right)\right) = \frac{1}{s}$ and hence $\lim_{k\to\infty}\psi\left(M(x_{m_k}, x_{n_k-2})\right) = 0$. Now, the continuity of ψ and the condition $\psi(0) = 0$ imply that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k-2}) = 0. \tag{3.15}$$

Also, b-quadrilateral inequality implies that

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq s\{d(x_{m_k}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})\}.$$

Applying $k \to \infty$ and using (3.12), we get $\epsilon = 0$ which is a contradiction. Thus $\{x_n\}$ is Cauchy sequence. The completeness of space X implies that there exists $x^* \in X$ such that $x_n \to x^* \in X$. We assert that $x^* \in X$ is a fixed point of T. Using b-quadrilateral inequality, we can write

$$d(x^*, Tx^*) \leq s\{d(x^*, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx^*)\}.$$

As T is generalized ψ_s-Geraghty contraction, therefore by applying ψ, we can have

$$\psi\big(d(x^*,Tx^*) - sd(x^*,x_{n+1}) - sd(x_{n+1},x_{n+2})\big) \le \psi(sd(Tx_{n+1},Tx^*))$$
$$\le \beta\left(\psi\big(M(x_{n+1},x^*)\big)\right)\psi\big(M(x_{n+1},x^*)\big), \quad (3.16)$$

where $M(x_{n+1},x^*) = \max\{d(x_{n+1},x^*), d(x_{n+1},x_{n+2}), d(x^*,Tx^*)\}$. Taking the upper limit as $n \to \infty$ and using (3.12) and the continuity of ψ, we obtain from (3.16)

$$\limsup_{n\to\infty} \beta\left(\psi\big(M(x_{n+1},x^*)\big)\right) = \frac{1}{s}.$$

So, we have $\lim_{n\to\infty} \psi\big(M(x_{n+1},x^*)\big) = 0$ and hence $\psi\big(d(x^*,Tx^*)\big) = 0$. Thus x^* is a fixed point of T. It is easy to show that this fixed point is unique.

Remark 3.5 (1) For $s = 1$, b-generalized metric space reduces to generalized metric space of Branciari [2].Therefore, Theorem 12 and Corollary 26 of Asadi et al. [3] can be obtained as a particular case of Theorem 3.2 and Theorem 3.4 respectively.

(2) As b-metric space is a b-generalized metric space, therefore Theorem 3.4 is an improvement and extension of the Corollary 1 of Faraji et al. [4]. This is an improvement in the sense that we have relaxed the continuity of map T.

4. Consequences

Now, we establish a common fixed point theorem in b-generalized metric space which is a consequence of Theorem 3.4.

Haghi et al. [9] proved the following lemma.

Lemma 4.1 Let X be a nonempty set and $f : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one to one.

Theorem 4.2 Let (X,d) be a b-generalized metric space and let $T,f: X \to X$ be two self-maps such that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subset of X. If there exist two functions $\beta \in \mathcal{F}$ and $\psi: [0,\infty) \to [0,\infty)$ is non-decreasing, continuous and $\psi(0) = 0$ such that

$$\psi\big(sd(Tx,Ty)\big) \le \beta\left(\psi\big(M(fx,fy)\big)\right)\psi(M(fx,fy)) \quad (4.1)$$

holds for all $x,y \in X$, where $M(fx,fy) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\}$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof: By Lemma 4.1, there exists $E \subseteq X$ such that $f(E) = f(X)$ and $f: E \to X$ is one to one. We define a map $g: f(E) \to f(E)$ by $g(fx) = Tx$. Clearly, g is well defined since f is one to one. Now, using (3.17), we have

$$\psi\left(d\big(g(fx), g(fy)\big)\right) \le \psi\big(sd(Tx,Ty)\big)$$
$$\le \beta\left(\psi\big(M(fx,fy)\big)\right)\psi\big(M(fx,fy)\big)$$

for all $fx, fy \in f(E)$. Since $f(E) = f(X)$ is complete, therefore by Theorem 3.4, there exists $z \in X$ such that $g(fz) = fz$, which implies $Tz = fz$. Hence, T and f have a coincidence point. Again, if w is another coincidence point of T and f such that $z \neq w$, then by (4.1)

$$\psi\big(d(Tw,Tz)\big) \leq \psi\big(sd(Tw,Tz)\big) \leq \beta\left(\psi\big(M(fw,fz)\big)\right)\psi\big(M(fw,fz)\big), \tag{4.2}$$

where

$$M(fw,fz) = \max\{d(fw,fz), d(fw,Tw), d(fz,Tz)\} = d(fw,fz) = d(Tw,Tz).$$

Since $\beta \in \mathcal{F}$, it follows from (4.2)

$$\psi\big(d(Tw,Tz)\big) < \psi\big(d(Tw,Tz)\big),$$

which is a contradiction. Hence z is a unique coincidence point of T and f. It is clear that T and f have a unique common fixed point whenever T and f are weakly compatible.

5. Conclusions

Here**,** we define the notion of $(\alpha\text{-}\psi)_s$-Geraghty Contractions in the set up of b-generalized metric space by inspiring to the idea of Asadi et al. [3] and obtain some new results as an extension of already existing fixed point as well as common fixed point theorems. More specifically, we work to improve and generalize recent fixed point results in b-metric space and generalized metric space established by Faraji et al. [3] and Asadi et al. [4] respectively.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publications of this paper.

Authors' Contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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