New Homeomorphisms

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1 Introduction

The study of generalized closed (g-closed) sets in a topological space was initiated by Levine (1963) and the concept of $T_{1/2}$ -spaces was introduced. Dunham (1977, 1982) further investigated the properties of $T_{1/2}$-spaces and defined a new closure operator cl* using generalized closed sets. Again Levine (1970) introduced the concept of semi-open sets and semi-continuity in a topological space. Further, Bhattacharrya and Lahiri introduced a new class of semi-generalized open sets by means of semi-open sets introduced by Levine (1963). In continuation of the previous studies, Balachandran *et al* (1991) introduced the concept of generalized continuous maps and generalized homeomorphism in a topological space.

In this chapter, let us introduce a new class of open sets namely $\mathcal{BS}$-g**-open sets and investigate some of their properties. Further, we introduce the concept of $\mathcal{BS}$-g**-continuous maps which includes the class of continuous maps in a bitopological space. Also we introduce $\mathcal{BS}$-g**-irresolute maps analogy to irresolute maps in a topological space and investigate some of their properties. Finally we introduce a new class of maps namely $\mathcal{BS}$-g**-homeomorphisms in a bitopological space and study some of their properties.

2 New Definitions

Definition 2.1. *Let M be a subset of X. Then M is said to be*

[(i)]

1. $\mathcal{BS}$*-g-closed if and only if* $\mathcal{BS}$*-cl(M)* $\subseteq F$ *whenever* $M \subseteq F$ *and F is* $\mathcal{BS}$*-open in X;*
2. $\mathcal{BS}$*-g-open if and only if* $X\backslash M$ *is* $\mathcal{BS}$*-g-closed*
3. $\mathcal{BS}$*-semi-open if and only if there exists an* $\mathcal{BS}$*-open set F such that* $F \subseteq M \subseteq$ $\mathcal{BS}$*-cl(F).*

Definition 2.2. *A map f:* $X \to Y$ *is called*

[(i)]$\mathcal{BS}$*-g-continuous if* $f^{-1}(V)$ *is* $\mathcal{BS}$*-g-open whenever V is* $\mathcal{BS}$*-open in Y;* $\mathcal{BS}$*-gc-irresolute if the inverse image of every* $\mathcal{BS}$*-g-closed set in Y is* $\mathcal{BS}$*-g-closed in X.*

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Theorem 2.1. *If A is $\mathcal{BS}$-g-closed set in X and if f: X $\to$ Y is $\mathcal{BS}$-continuous and $\mathcal{BS}$-closed, then f(A) is $\mathcal{BS}$-g-closed.*

Theorem 2.2. *If f: X $\to$ Y is $\mathcal{BS}$-continuous and $\mathcal{BS}$-closed and if B is a $\mathcal{BS}$-g-closed ((or) $\mathcal{BS}$-g-open) subset of Y, then f^{-1}(B) is $\mathcal{BS}$-g-closed (or) ($\mathcal{BS}$-g-open) in X.*

3 Characterizations of $\mathcal{BS}$-g**-open sets

Definition 3.1. *Let M be a subset of X. Then $\mathcal{BS}$-cl**(M) = $\cap$ {F/M $\subseteq$ F and F is $\mathcal{BS}$-g-closed}.*

Remark 3.1. *$\mathcal{BS}$-cl**(M) is a Kuratowki closure operator on X.*

Definition 3.2. *Let M be a subset of X. Then M is said to be $\mathcal{BS}$-g**-open if and only if there exists an $\mathcal{BS}$-open set U of X such that U $\subseteq$ M $\subseteq$ $\mathcal{BS}$-cl**(U).*

If X = $\{\alpha, \beta, \gamma\}$, $\tau_1 = \{\varphi$, X, $\{\alpha\}\}$ and $\tau_2 = \{\varphi$, X, $\{\alpha, \beta\}\}$, then $\mathcal{BS}$-g**-open sets of (X, τ_1, τ_2) are $\{\varphi$, X, $\{\alpha\}$, $\{\alpha, \beta\}$, $\{\alpha, \gamma\}\}$.

Remark 3.2. *If S and T are subsets of X such that S $\subseteq$ T, then $\mathcal{BS}$-cl**(S) $\subseteq$ $\mathcal{BS}$-cl**(T).*

Theorem 3.1. *Let M be a subset of X. Then M is $\mathcal{BS}$-g**-open in X if and only if M $\subseteq$ $\mathcal{BS}$-cl** ($\mathcal{BS}$-int(M)).*

2. *Proof.* If M is a $\mathcal{BS}$-g**-open set of X, then there exists an $\mathcal{BS}$-open set U such that U $\subseteq$ M $\subseteq$ $\mathcal{BS}$-cl** (U). U $\subseteq$ M implies U $\subseteq$ $\mathcal{BS}$-int(M). Hence by Remark 3.2, $\mathcal{BS}$-cl**(U) $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(M)). Therefore M $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(M)). Conversely, let M $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(M)). Let U = $\mathcal{BS}$-int(M). Then U $\subseteq$ M $\subseteq$ $\mathcal{BS}$-cl**(U). Hence M is a $\mathcal{BS}$-g**-open set in X.

If M is an $\mathcal{BS}$-open set in X, then M is a $\mathcal{BS}$-g**-open.

Proof. Let M be a $\mathcal{BS}$-open set in X. It implies M = $\mathcal{BS}$-int(M) $\subseteq$ $\mathcal{BS}$-cl** ($\mathcal{BS}$-int(M)). Hence M is a $\mathcal{BS}$-g**-open in X.

The converse of Proposition 3 need not be true.

Refer Example **??**, clearly M = $\{\alpha, \gamma\}$ is a $\mathcal{BS}$-g**-open set in X but it is not $\mathcal{BS}$-open in X.

Definition 3.3. *A bitopological space X is said to be $\mathcal{BS}$-$T_{1/2}$ space if every $\mathcal{BS}$-g-open set of X is $\mathcal{BS}$-open in X.*

Definition 3.4. *A bitopological space X is said to be $\mathcal{BS}$-g**-$T_{1/2}$-space if every $\mathcal{BS}$-g**-open set of X is $\mathcal{BS}$-open in X.*

Remark 3.3. *In $\mathcal{BS}$-$T_{1/2}$ space, every $\mathcal{BS}$-semi-open set is a $\mathcal{BS}$-g**-open.*

Remark 3.4. *$\mathcal{BS}$-cl**(D) $\subseteq$ $\mathcal{BS}$-cl(D) if D is a subset of X.*

Theorem 3.2. *If E is a $\mathcal{BS}$-g**-open set in X, then E is $\mathcal{BS}$-semi-open in X.*

Proof. Given E is $\mathcal{BS}$-g**-open set in X. Therefore there exists $\mathcal{BS}$-open set U such that U $\subseteq$ E $\subseteq$ $\mathcal{BS}$-cl**(U). By Remark 3.4 $\mathcal{BS}$-cl**(U) $\subseteq$ $\mathcal{BS}$-cl(U). Hence U $\subseteq$ E $\subseteq$ $\mathcal{BS}$-cl(U) implies E is $\mathcal{BS}$-semi-open.

The converse of Theorem 3.2 need not be true.

Let X = $\{\alpha,\beta,\gamma\}$, τ_1 = $\{\varphi$, X, $\{\alpha\}\}$ and τ_2 = $\{\varphi$, X$\}$. Then E = $\{\alpha,\beta\}$ is a $\mathcal{BS}$-semi-open set but it is not a $\mathcal{BS}$-g**-open.

Remark 3.5. *If P and Q are subsets of X, then $\mathcal{BS}$-cl**(P $\cup$ Q) = $\mathcal{BS}$-cl**(P) $\cup$ $\mathcal{BS}$-cl**(Q).*

Theorem 3.3. *If M and N are $\mathcal{BS}$-g**-open sets of X, then M $\cup$ N is also a $\mathcal{BS}$-g**-open set in X.*

Proof. Given M and N are $\mathcal{BS}$-g**-open sets in X. Then there exists $\mathcal{BS}$-open sets U and V respectively such that U $\subseteq$ M $\subseteq$ $\mathcal{BS}$-cl**(U) and V $\subseteq$ N $\subseteq$ $\mathcal{BS}$-cl**(V). By Remark 3.5, $\mathcal{BS}$-cl**(U) $\cup$ $\mathcal{BS}$-cl**(V) = $\mathcal{BS}$-cl**(U $\cup$ V). Hence U $\cup$ V $\subseteq$ M $\cup$ N $\subseteq$ $\mathcal{BS}$-cl**(U $\cup$ V). Hence M $\cup$ N is also $\mathcal{BS}$-g**-open set in X.

If M and N are $\mathcal{BS}$-g**-open in X, then M $\cap$ N need not be $\mathcal{BS}$-g**-open in X.

Let X = $\{\alpha,\beta,\gamma\}$, τ_1 = $\{\varphi$, X, $\{\alpha\}\}$ and τ_2 = $\{\varphi$, X, $\{\beta\}\}$. Then M = $\{\alpha,\gamma\}$ and N = $\{\beta,\gamma\}$ are $\mathcal{BS}$-g**-open sets in X and M $\cap$ N = $\{\gamma\}$ is not a $\mathcal{BS}$-g**-open set in X.

Theorem 3.4. *Let P be a $\mathcal{BS}$-g**-open set in X and Q be any set such that P $\subseteq$ Q $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(P)). Then Q is a $\mathcal{BS}$-g**-open set in X.*

Proof. Given P is a $\mathcal{BS}$-g**-open set in X. Therefore by Theorem 3.1, P $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(P)). P $\subseteq$ Q implies $\mathcal{BS}$-int(P) $\subseteq$ $\mathcal{BS}$-int(Q), hence, $\mathcal{BS}$-cl** ($\mathcal{BS}$-int(P)) $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(Q)). Therefore Q $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(P)) $\subseteq$ $\mathcal{BS}$-cl**($\mathcal{BS}$-int(Q)). Thus, Q is a $\mathcal{BS}$-g**-open set in X.

Remark 3.6. *Let P $\subseteq$ X. If f: X $\to$ Y is $\mathcal{BS}$-g-continuous map, then f($\mathcal{BS}$-cl**(P)) $\subseteq$ $\mathcal{BS}$-cl(f(P)).*

Theorem 3.5. *Let f: X $\to$ Y be $\mathcal{BS}$-g-continuous $\mathcal{BS}$-open map. If P is a $\mathcal{BS}$-g**-open set in X, then f(P) is a $\mathcal{BS}$-semi-open set in Y.*

Proof. Given P is $\mathcal{BS}$-g*-open set in X. Therefore there exist an $\mathcal{BS}$-open set U such that U $\subseteq$ P $\subseteq$ $\mathcal{BS}$-cl**(U). By Remark 3.6, we have f($\mathcal{BS}$-cl**(P)) $\subseteq$ $\mathcal{BS}$-cl(f(P)). Hence f(U) $\subseteq$ f(P) $\subseteq$ f($\mathcal{BS}$-cl**(U)) $\subseteq$ $\mathcal{BS}$-cl(f(U)). Since f is an $\mathcal{BS}$-open map, f(U) is $\mathcal{BS}$-open in Y. This implies f(P) is a $\mathcal{BS}$-semi-open set in Y.

Theorem 3.6. *Let f: X $\to$ Y be a $\mathcal{BS}$-homeomorphism. If P is a $\mathcal{BS}$-g**-open set in X, then f(P) is $\mathcal{BS}$-g**-open set in Y.*

Proof. Given P is a $\mathcal{BS}$-g**-open set in X. Therefore there exists an $\mathcal{BS}$-open set U such that U $\subseteq$ P $\subseteq$ $\mathcal{BS}$-cl**(U), implies f(U) $\subseteq$ f(P) $\subseteq$ f($\mathcal{BS}$-cl**(U)). Since f is $\mathcal{BS}$-homeomorphism and by Theorem 2.2, we have f($\mathcal{BS}$-cl**(U)) $\subseteq$ $\mathcal{BS}$-cl**(f(U)). Therefore f(U) $\subseteq$ f(P) $\subseteq$ $\mathcal{BS}$-cl**(f(U)) and hence f(P) is a $\mathcal{BS}$-g**-open set in Y.

Theorem 3.7. *Let f: $X \to Y$ be a $\mathcal{BS}$-homeomorphism. If M is a $\mathcal{BS}$-g**-open set in Y, then f^{-1}(M) is $\mathcal{BS}$-g**-open set in X.*

Proof. Given M is a $\mathcal{BS}$-g**-open set in Y. Therefore there exists an $\mathcal{BS}$-open set U such that U ⊆ M ⊆ $\mathcal{BS}$-cl**(U), implies f^{-1}(U) ⊆ f^{-1}(M) ⊆ f^{-1}($\mathcal{BS}$-cl**(U)). Since f is $\mathcal{BS}$-homeomorphism and by Theorem 2.2, we f^{-1}($\mathcal{BS}$-cl**(U)) ⊆ $\mathcal{BS}$-cl**(f^{-1}(U)). Therefore f^{-1}(U) ⊆ f^{-1}(M) ⊆ $\mathcal{BS}$-cl**(f^{-1}(U)) and hence f^{-1}(M) is $\mathcal{BS}$-g**-open in X.

Definition 3.5. *A subset T of X is said to be $\mathcal{BS}$-g**-closed if and only if $X \backslash T$ is $\mathcal{BS}$-g**-open.*

Definition 3.6. *Let T be a subset of X. Then the $\mathcal{BS}$-g**-closure of T is defined as $\mathcal{BS}$-g**-cl(T) = ∩{F: T ⊆ F and F is $\mathcal{BS}$-g**-closed}.*

Remark 3.7. *From the above definition, $\mathcal{BS}$-g**-cl(T) is the smallest $\mathcal{BS}$-g**-closed set containing T.*

Definition 3.7. *Let T be a subset of X. Let $x \in X$. Then x is said to be a $\mathcal{BS}$-g**-limit point of P if and only if every $\mathcal{BS}$-g**-open set containing x contains at least one point other than x.*

Definition 3.8. *Let T be a subset of X. Then the set of all $\mathcal{BS}$-g**-limit points of T is said to be $\mathcal{BS}$-g**-derived set of T and it is denoted by $\mathcal{BS}$-Dg**(T).*

Theorem 3.8. *Let P be a subset of X. Then $x \in$ $\mathcal{BS}$-g**-cl(P) if and only if every $\mathcal{BS}$-g**-open set U contains x intersect with P.*

Proof. We prove this theorem in contra positive way. If x ∉ $\mathcal{BS}$-g**-cl(P), then x ∈ X\\$\mathcal{BS}$-g**-cl(P). Let U = X\\$\mathcal{BS}$-g**-cl(P). Then by Remark 3.7 U is $\mathcal{BS}$-g**-open set which does not intersect with P. Conversely, if U is a $\mathcal{BS}$-g**-open set of x which does not intersect with P, then X\U is a $\mathcal{BS}$-g**-closed set containing P. This implies x ∉ $\mathcal{BS}$-g**-cl(P).

Theorem 3.9. *Let M be a subset of X. Let $\mathcal{BS}$-Dg**(M) be the set of all $\mathcal{BS}$-g**-limit points of M. Then $\mathcal{BS}$-g**-cl(M) = M ∪ $\mathcal{BS}$-Dg**(M).*

Proof. Let x ∈ M ∪ $\mathcal{BS}$-Dg**(M). This implies either x ∈ M or x ∈ $\mathcal{BS}$-Dg**(M). If x ∈ M, then x ∈ $\mathcal{BS}$-g**-cl(M). If x ∈ $\mathcal{BS}$-Dg**(M), then every $\mathcal{BS}$-g**-open set contains x will intersect with M. Therefore x ∈ $\mathcal{BS}$-g**-cl(M). This implies M ∪ $\mathcal{BS}$-Dg**(M) ⊆ $\mathcal{BS}$-g**-cl(M). If x ∈ $\mathcal{BS}$-g**-cl(M), then to prove x ∈ M ∪ $\mathcal{BS}$-Dg**(M). If x ∈ M, then x ∈ M ∪ $\mathcal{BS}$-Dg**(M). If x ∉ M, since x ∈ $\mathcal{BS}$-g**-cl(M) implies every $\mathcal{BS}$-g**-open set of x intersects with M. Hence x ∈ $\mathcal{BS}$-Dg**(M). Therefore $\mathcal{BS}$-g**-cl(M) = M ∪ $\mathcal{BS}$-Dg**(M).

4 Properties of $\mathcal{BS}$-g**-Continuous Maps

Definition 4.1. *A map f: X → Y is said to be $\mathcal{BS}$-g**-continuous if the inverse image of every $\mathcal{BS}$-open set in Y is $\mathcal{BS}$-g**-open in X.*

Theorem 4.1. *If f: X → Y is a $\mathcal{BS}$-continuous map, then it is $\mathcal{BS}$-g**-continuous.*

Proof. Let U be an $\mathcal{BS}$-open set in Y. Since f is $\mathcal{BS}$-continuous, f^{-1}(U) is $\mathcal{BS}$-open in X. By Proposition 3, f^{-1}(U) is $\mathcal{BS}$-g**-open in X. Hence f is $\mathcal{BS}$-g**-continuous.

The converse of Theorem 4.1 need not be true.

Let X = $\{\alpha, \beta, \gamma\}$, $\tau_1 = \{\varphi, X, \{\alpha\}\}$ and $\tau_2 = \{\varphi, X, \{\alpha, \beta\}\}$.

Let Y = {1, 2}, $\sigma_1 = \{\varphi, Y, \{1\}\}$ and $\sigma_2 = \{\varphi, Y\}$.

Let f: X → Y be a map defined by f(α) = f(γ) = 1, f(β) = 2.

Then f is $\mathcal{BS}$-g**-continuous but it is not a $\mathcal{BS}$-continuous map.

Remark 4.1. *In $\mathcal{BS}$-g**-$T_{1/2}$ space, the concepts of $\mathcal{BS}$-continuity and $\mathcal{BS}$-g**-continuity coincide.*

Theorem 4.2. *Let f: X → Y be a map. Then the following statements are equivalent.*

*[(i)]f is $\mathcal{BS}$-g**-continuous. the inverse image of each $\mathcal{BS}$-closed set in Y is $\mathcal{BS}$-g**-closed in X.*

2. *Proof.* (i) ⇒ (ii) Let C be any $\mathcal{BS}$-closed set in Y. Then Y\C is $\mathcal{BS}$-open in Y. Since f is $\mathcal{BS}$-g**-continuous, f^{-1}(Y\C) is $\mathcal{BS}$-g**-open in X. Therefore X\f^{-1}(C) is $\mathcal{BS}$-g**-open in X, which implies f^{-1}(C) is $\mathcal{BS}$-g**-closed in X.

(ii) ⇒ (i) Let D be an $\mathcal{BS}$-open set in Y. Then Y\D is $\mathcal{BS}$-open in Y. This implies f^{-1}(Y\D) is $\mathcal{BS}$-g**-closed in X, which implies X\f^{-1}(D) is $\mathcal{BS}$-g**-closed in X. Therefore f^{-1}(U) is $\mathcal{BS}$-g**-open in X. Hence f is $\mathcal{BS}$-g**-continuous.

Theorem 4.3. *If f: X → Y is a $\mathcal{BS}$-g**-continuous map, then f($\mathcal{BS}$-g**-cl(C)) ⊆ $\mathcal{BS}$-cl(f(C)).*

Proof. Since f(C) ⊆ $\mathcal{BS}$-cl(f(C)), implies C ⊆ f^{-1}($\mathcal{BS}$-cl(f(C))). Then $\mathcal{BS}$-cl(f(C)) is a $\mathcal{BS}$-closed set in Y and f is $\mathcal{BS}$-g**-continuous map implies f^{-1}($\mathcal{BS}$-cl(f(C))) is $\mathcal{BS}$-g**-closed in X. Hence $\mathcal{BS}$-g**-cl(C) ⊆ f^{-1}($\mathcal{BS}$-cl(f(C))). Therefore f($\mathcal{BS}$-g**-cl(C)) ⊆ $\mathcal{BS}$-cl(f(C)).

5 Relation of $\mathcal{BS}$-g**-Continuous Maps with $\mathcal{BS}$-g**-Irresolute Maps

Definition 5.1. *Let f: X → Y be a map. Then it is called $\mathcal{BS}$-g**-irresolute if the inverse image of every $\mathcal{BS}$-g**-open set of Y is $\mathcal{BS}$-g**-open in X.*

Remark 5.1. *A map f: X → Y is a $\mathcal{BS}$-g**-irresolute map if and only if the inverse image of every $\mathcal{BS}$-g**-closed set in Y is $\mathcal{BS}$-g**-closed in X.*

Theorem 5.1. *If f: X → Y is a $\mathcal{BS}$-g**-irresolute, then f is $\mathcal{BS}$-g**-continuous map.*

Proof. Let F be an $\mathcal{BS}$-open set in X. Since f is $\mathcal{BS}$-g**-irresolute map, implies $f^{-1}(F)$ is $\mathcal{BS}$-g**-open in X. Hence f is $\mathcal{BS}$-g**-continuous..

The converse of Theorem 5.1 need not be true.

Let $X = \{\alpha, \beta, \gamma\}$, $Y = \{1, 2, 3\}$, $\tau_1 = \{\varphi, X, \{\alpha, \beta\}\}$, $\tau_2 = \{\varphi, X, \{\beta\}, \{\gamma\}, \{\beta, \gamma\}\}$, $\sigma_1 = \{\varphi, Y, \{1\}\}$ and $\sigma_2 = \{\varphi, Y, \{2\}\}$. Define f: $X \to Y$ by $f(\alpha) = 3$; $f(\beta) = 2$; $f(\gamma) = 1$. Clearly f is $\mathcal{BS}$-g**-continuous but it is not $\mathcal{BS}$-g**-irresolute.

Theorem 5.2. *Let f: $X \to Y$ be a $\mathcal{BS}$-continuous map and Y is $\mathcal{BS}$-g**-$T_{1/2}$-space then f is $\mathcal{BS}$-g**-irresolute.*

Proof. Let C be $\mathcal{BS}$-g**-open set in Y. Since Y is $\mathcal{BS}$-g**-$T_{1/2}$ space, implies C is an $\mathcal{BS}$-open set in Y. Since f is $\mathcal{BS}$-continuous, implies $f^{-1}(C)$ is $\mathcal{BS}$-g**-open in X. Therefore f is $\mathcal{BS}$-g**-irresolute map.

Theorem 5.3. *Let f: $X \to Y$ be a $\mathcal{BS}$-g**-irresolute map and g: $Y \to Z$ be a $\mathcal{BS}$-g**-irresolute map. Then gof: $X \to Z$ is $\mathcal{BS}$-g**-irresolute.*

Proof. Let M be a $\mathcal{BS}$-g**-open set in Z. Then $g^{-1}(M)$ is $\mathcal{BS}$-g**-open in Y which implies $f^{-1}(g^{-1}(M))$ is $\mathcal{BS}$-g**-open in X. Therefore $(gof)^{-1}(M)$ is $\mathcal{BS}$-g**-open in X. Hence, gof is $\mathcal{BS}$-g**-irresolute.

6 $\mathcal{BS}$-g**-Connected Sets

Definition 6.1. *Y is said to be $\mathcal{BS}$-g**-connected if Y cannot be written as disjoint union of two non-empty $\mathcal{BS}$-g**-open sets.*

Theorem 6.1. *For a bitopological space Y, the following statements are equivalent.*

*[(i)]Y is $\mathcal{BS}$-g**-connected. The only subsets of Y which are both $\mathcal{BS}$-g**-open and $\mathcal{BS}$-g**-closed are φ and Y.*

2. *Proof.* i) ⇒ (ii) Let U be a $\mathcal{BS}$-g**-open and $\mathcal{BS}$-g**-closed subsets of Y. Then $Y \backslash U$ is both $\mathcal{BS}$-g**-open and $\mathcal{BS}$-g**-closed. Since Y is the disjoint union of $\mathcal{BS}$-g**-open set U and $Y \backslash U$, implies one of these must be empty, that is $U = \varphi$ or $Y \backslash U = \varphi$.

(ii) ⇒ (i) Suppose that $Y = M \cup N$ where M and N are disjoint non-empty $\mathcal{BS}$-g**-open set of X. Then $M(=Y \backslash N)$ is $\mathcal{BS}$-g**-closed. Hence M is both $\mathcal{BS}$-g**-open and $\mathcal{BS}$-g**-closed subset of Y. By assumption, $M = \varphi$ or $M = X$. This implies Y is $\mathcal{BS}$-g**-connected.

Theorem 6.2. *[(i)]*

1. *If f: $X \to Y$ is a $\mathcal{BS}$-g**-continuous surjection map and X is $\mathcal{BS}$-g**-connected, then Y is $\mathcal{BS}$-connected.*

2. *If f: $X \to Y$ is a $\mathcal{BS}$-g**-irresolute surjection map and X is $\mathcal{BS}$-g**-connected, then Y is $\mathcal{BS}$-g**-connected.*

Proof. (i) Suppose that Y is not $\mathcal{BS}$-connected. Then Y = C ∪ D, where C and D are disjoint non-empty $\mathcal{BS}$-open sets in Y. Since f is $\mathcal{BS}$-g**-continuous and onto, X = f^{-1}(C) ∪ f^{-1}(D) where f^{-1}(C) and f^{-1}(D) are disjoint non-empty $\mathcal{BS}$-g**-open sets which is a contradiction to our assumption that X is $\mathcal{BS}$-g**-connected. Hence Y is $\mathcal{BS}$-connected. (ii) It follows from Definition 5.1

7 $\mathcal{BS}$-g**-Homeomorphisms

Definition 7.1. *A map f: X → Y is said to be $\mathcal{BS}$-g**-open map if f(M) is $\mathcal{BS}$-g**-open in Y for every $\mathcal{BS}$-open set M in X.*

Theorem 7.1. *If f: X → Y is an $\mathcal{BS}$-open map, then it is an $\mathcal{BS}$-g**-open map.*

Proof. Given f: X → Y is an $\mathcal{BS}$-open map. Let H be any $\mathcal{BS}$-open set in X. Then f(H) is $\mathcal{BS}$-open in Y. By Proposition 3, f(H) is $\mathcal{BS}$-g**-open in Y. Hence f is an $\mathcal{BS}$-g**-open map.

The converse of Theorem 7.1 need not be true.

Let X = Y = $\{\alpha, \beta, \gamma\}$, $\tau_1 = \{\varphi$, X, $\{\alpha\}\}$ and $\tau_2 = \{\varphi$, X, $\{\beta\}\}$.

Let $\sigma_1 = \{\varphi$, Y, $\{\alpha\}\}$ and $\sigma_2 = \{\varphi$, Y, $\{\alpha, \beta\}\}$.

Then define f: X → Y as f(α) = α, f(β) = γ and f(γ) = β. Clearly f is $\mathcal{BS}$-g**-open map but it is not an $\mathcal{BS}$-open map.

Definition 7.2. *A map f: X → Y is said to be $\mathcal{BS}$-g**-closed if f(H) is $\mathcal{BS}$-g**-closed in Y for every $\mathcal{BS}$-closed set H in X.*

Remark 7.1. *If f: X → Y is a $\mathcal{BS}$-closed map, then it is a $\mathcal{BS}$-g**-closed map and the converse need not be true.*

Proof. Similar to the proof of Theorem 7.1.

Definition 7.3. *A bijection map f: X → Y is called $\mathcal{BS}$-g**-homeomorphism if f is both $\mathcal{BS}$-g**-continuous and $\mathcal{BS}$-g**-open.*

Remark 7.2. *Every $\mathcal{BS}$-homeomorphism is $\mathcal{BS}$-g**-homeomorphism but not conversely.*

Theorem 7.2. *For any bijection map f: X → Y, the following statements are equivalent.*

*[(i)]Its inverse map f^{-1}: Y → X is $\mathcal{BS}$-g**-continuous. f is a $\mathcal{BS}$-g**-open map. f is a $\mathcal{BS}$-g**-closed map.*

3. *Proof.* $(i) \Rightarrow$ (ii) Let I be any $\mathcal{BS}$-open set in X. Since f^{-1} is $\mathcal{BS}$-g**-continuous, the inverse image of I under f^{-1} namely f(I) is $\mathcal{BS}$-g**-open in Y. Hence f is $\mathcal{BS}$-g**-open map. $(ii) \Rightarrow (iii)$ Let J be any $\mathcal{BS}$-closed set in X. Then X\J is $\mathcal{BS}$-open in X. Since f is $\mathcal{BS}$-g**-open map, f(X\J) is $\mathcal{BS}$-g**-open in Y. But f(X\J) = Y\f(J), implies f(J) is $\mathcal{BS}$-g**-closed in Y. Therefore f is $\mathcal{BS}$-g**-closed map. $(iii) \Rightarrow (i)$ Let J be any $\mathcal{BS}$-closed set in X. Then the inverse image of J under f^{-1}, namely f(J) is $\mathcal{BS}$-g**-closed in Y. Since f is a $\mathcal{BS}$-g**-closed map, f^{-1} is $\mathcal{BS}$-g**-continuous.

Theorem 7.3. *Let h: $X \to Y$ be bijective and $\mathcal{BS}$-g**-continuous map. Then the following statements are equivalent.*

*[(i)]h is $\mathcal{BS}$-g**-open map. h is $\mathcal{BS}$-g**-homeomorphism. h is $\mathcal{BS}$-g**-closed map.*

3. *Proof.* $(i) \Rightarrow (ii)$ By assumption, h is bijective, $\mathcal{BS}$-g**-continuous and $\mathcal{BS}$-g**-open map. Then by definition, h is a $\mathcal{BS}$-g**-homeomorphism. $(ii) \Rightarrow (iii)$ By assumption, h is $\mathcal{BS}$-g**-open and bijective. By Theorem 7.2, h is $\mathcal{BS}$-g**-closed map. $(iii) \Rightarrow (i)$ By assumption, h is $\mathcal{BS}$-g**-closed and bijective. By Theorem 7.2, h is a $\mathcal{BS}$-g**-open map.

CONCLUSION

Topology and Bitopology are used in several areas such as quantum field theory, image processing, molecular biology and cosmology and they can also be used to describe the overall shape of the universe. The various possible positions of a robot can be described by a manifold called configuration space. In the area of motion planning, one finds paths between two points in configuration space.

General topology is important in many fields of applied sciences as well as branches of mathematics. In reality it is used in data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, information systems, particle physics and quantum physics etc.

The notions of sets and functions in topological spaces, bitopological spaces, generalized topological spaces, minimal spaces and ideal minimal spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences.

By researching generalizations of closed sets in various fields in general topology, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, all concepts defined in this thesis will have many possibilities of applications in digital topology and computer graphics.

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